NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1433

STABILITY OF CYLINDRICAL AND CONICAL SHELLS OF CIRCULAR CROSS SECTION, WITH SIMULTANEOUS ACTION OF AXIAL COMPRESSION AND EXTERNAL NORMAL PRESSURE

By Kh. M. Mushtari and A. V. Sachenkov

Translation of "Ob ustoǐchivosti tsilindricheskikh i konicheskikh obolochek krugovogo cechenia pri sovmestnom deǐstvii osevogo szhatiia i vneshnego normalnogo davleniia." Prikladnaia Matematika i Mekhanika, vol. 18, no. 6, November-December 1954.

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We consider in this report the determination of the upper limit of critical loads in the case of simultaneous action of a compressive force, uniformly distributed over plane cross sections, and of isotropic external normal pressure on cylindrical or conical shells of circular cross section. As a starting point we use the differential equations for neutral equilibrium of conical shells (ref. 1) which have been used for the solution of the problem of stability of conical shells under torsion and under axial compression (ref. 2); upon solution of the problem it is possible to satisfy all boundary conditions, in contrast to the report (ref. 3) where no attention is paid to the fulfillment of the boundary conditions and to the report (ref. 4) where only part of the boundary conditions are satisfied by solution of the problem according to Galerkin's method. Approximate formulas are used for the determination of the critical external normal pressure with simultaneous action of longitudinal compression. Let us note that the formulas suggested in reference 5 are not well founded and may lead, in a number of cases, to a substantial mistake in the magnitude of the critical load.

1. SYMBOLS

We shall use the following symbols:

2γ angle of taper

r distance, along the generatrix, between the vertex of the cone and a point on the median surface

r_0 distance to the smaller one of the bases

*"Ob ustoǐchivosti tsilindricheskikh i konicheskikh obolochek krugovogo cecheniia pri sovmestnom deǐstvii osevogo szhatiia i vneshnego normalnogo davleniia." Prikladnaia Matematika i Mekhanika, vol. 18, no. 6, November-December 1954, pp. 667-674.

L	length of the shell along the generatrix
$2h$	thickness of the shell
φ	angle between the axial plane and the plane under consideration
n	number of waves forming along the periphery when the shell buckles
w	normal displacement of the point on the median surface during buckling
p_0	external normal pressure, acting both on the lateral surface and on the bases of the shell section considered
T_0	additional compressive force applied to the smaller of the end cross sections
T_{10}, T_{20}	linear membrane forces up to buckling, determined according to the momentless theory
T_1, T_2, S	additional membrane forces appearing during buckling
ϵ_2	additional annular expansion
E	modulus of elasticity
σ	lateral-expansion coefficient

$$\left.\begin{aligned}&K = \frac{2Eh}{1 - \sigma^2} \qquad D = \frac{2Eh^3}{3(1 - \sigma^2)} \text{ (rigidities of the shell)}\\ &\tau = 1 + \frac{L}{r_0} \qquad t = \ln \tau \qquad m_1 = \frac{m\pi}{t} \quad (m \text{ - integer})\end{aligned}\right\} \tag{1.1}$$

$$\left.\begin{aligned}&R_0 = r_0 \tan \gamma \qquad \nu = \frac{(1 - \sigma)}{2} \qquad \mu_1 = 1 - 2\nu^2\\ &\mu_2 = \nu^2 - 3\nu^4 \qquad \varphi_1 = \varphi \sin \gamma \qquad n_1 = \frac{n}{\sin \gamma}\end{aligned}\right\} \tag{1.2}$$

$$\epsilon^2 = \frac{\nu h^2\left(1 - \tau^{2\nu-2}\right)}{3r_0^2\cot^2\gamma\left(1 - \sigma^2\right)(1 - \nu)(\tau^{2\nu} - 1)} \qquad D' = \frac{(1 + 2\nu)Eh(\tau^{2\nu} - 1)}{\left[\nu r_0 \tan^3\gamma\left(\tau^{(1+2\nu)} - 1\right)\right]}$$

$$\lambda = \frac{T_0(1 + 2\nu)\left(1 - \tau^{(2\nu-1)}\right)}{p_0R_0(1 - 2\nu)\left(\tau^{(1+2\nu)} - 1\right)} \qquad \lambda_1 = \frac{0.5 + \lambda\left[m_1^2 + 0.25(1 + 2\nu)^2\right]}{m_1^2} \tag{1.3}$$

$$\eta^2 = \frac{\epsilon^2\left(m_1^2 + \nu^2\right)m_1^4}{\left[m_1^2 + (1 - \nu)^2\right]\left[m_1^4 + \mu_1m_1^2 + \mu_2\right]}$$

$$K' = \frac{D'\left[m_1^2 + 0.25(1 + 2\nu)^2\right]\left(m_1^4 + \mu_1m_1^2 + \mu_2\right)}{\left[m_1^4\left(m_1^2 + \nu^2\right)\right]} \tag{1.4}$$

$$\delta_1 = \sqrt[4]{3}\,\frac{1}{\sqrt{\eta}}$$

2. FULFILLMENT OF THE BOUNDARY CONDITIONS AND INTEGRATION OF THE EQUATION OF COMBINED DEFORMATION

The differential equations after introduction of the stress function will be (ref. 1)

$$Dr\Delta\Delta w - \cot\gamma\,\frac{\partial^2 f}{\partial r^2} - r\left[T_{10}\,\frac{\partial^2 w}{\partial r^2} + T_{20}\left(\frac{1}{r}\,\frac{\partial w}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2 w}{\partial \varphi_1^2}\right)\right] = 0 \tag{2.1}$$

$$r\Delta\Delta f + 2Eh\cot\gamma\,\frac{\partial^2 w}{\partial r^2} = 0$$

where

$$\Delta(\ ) = \frac{\partial^2(\ )}{\partial r^2} + \frac{1}{r}\frac{\partial(\ )}{\partial r} + \frac{1}{r^2}\frac{\partial^2(\ )}{\partial \varphi_1{}^2}$$

$$T_1 = \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \varphi_1{}^2} \qquad T_2 = \frac{\partial^2 f}{\partial r^2} \qquad S = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial f}{\partial \varphi_1}\right) \tag{2.2}$$

$$T_{10} = -\frac{p_0 r \tan \gamma}{2} - \frac{T_0 r_0}{r} \qquad T_{20} = -p_0 r \tan \gamma \tag{2.3}$$

If the ends of the shell are simply supported, the conditions

$$w = 0 \qquad \frac{\partial^2 w}{\partial r^2} + \frac{\sigma}{r}\left(\frac{\partial w}{\partial r} + \frac{1}{r}\frac{\partial^2 w}{\partial \varphi_1{}^2}\right) = 0 \tag{2.4}$$

$$T_1 = 0 \qquad \epsilon_2 = 0$$

must be satisfied on the edges $r = r_0$ and $r = r_0 + L$.

We introduce the following substitutions:

$$z = \ln \frac{r}{r_0} \qquad f = F \cos n_1\varphi_1 \qquad w = e^{\nu z} w_1 \cos n_1\varphi_1 \qquad t = \ln\left(1 + \frac{L}{r_0}\right) \tag{2.5}$$

Then, taking equations (2.3) into consideration, we bring equations (2.1) into the form

$$\frac{d^4F}{dz^4} - 4\frac{d^3F}{dz^3} + \left(4 - 2n_1{}^2\right)\frac{d^2F}{dz^2} + 4n_1{}^2\frac{dF}{dz} + \left(n_1{}^4 - 4n_1{}^2\right)F +$$
$$2Ehr_0 \cot \gamma e^{(1+\nu)z}\left[\frac{d^2w_1}{dz^2} + (2\nu - 1)\frac{dw_1}{dz} + \left(\nu^2 - \nu\right)w_1\right] = 0$$

$$
\begin{aligned}
&e^{(\nu-4)z}\left\{\frac{d^4w_1}{dz^4} - 4(1-\nu)\frac{d^3w_1}{dz^3} + K_{1n}\frac{d^2w_1}{dz^2} + K_{2n}\frac{dw_1}{dz} + K_{3n}w_1 - \right. \\
&\frac{r_0 \cot\gamma}{D} e^{(1-\nu)z}\left(\frac{d^2F}{dz^2} - \frac{dF}{dz}\right) + \frac{p_0 \tan\gamma}{D} r_0^3 e^{3z}\left[\frac{1}{2}\frac{d^2w_1}{dz^2} + \left(\nu + \frac{1}{2}\right)\frac{dw_1}{dz} - K_{4n}w_1\right] + \\
&\left.\frac{T_0 r_0^2}{D} e^z\left[\frac{d^2w_1}{dz^2} + (2\nu - 1)\frac{dw_1}{dz} + \left(\nu^2 - \nu\right)w_1\right]\right\} = 0 \qquad (2.6)
\end{aligned}
$$

where

$$
\left.
\begin{aligned}
K_{1n} &= 6\nu^2 - 12\nu + 4 - 2n_1^2 \\
K_{2n} &= 4\nu^3 - 12\nu^2 + 8\nu + 4n_1^2(1 - \nu) \\
K_{3n} &= \nu^4 - 4\nu^3 + 4\nu^2 - n_1^2\left(4 - 4\nu + 2\nu^2\right) + n_1^4 \\
K_{4n} &= n_1^2 - \frac{1}{2}\nu - \frac{1}{2}\nu^2
\end{aligned}
\right\} \qquad (2.7)
$$

The boundary conditions (2.4) assume the form

$$
w_1 = 0 \qquad \frac{d^2w_1}{dz^2} = 0 \qquad (2.8)
$$

$$
\frac{dF}{dz} - n_1^2F = 0 \qquad \frac{d^2F}{dz^2} - \frac{dF}{dz} = 0 \qquad \text{For } z = 0 \text{ and } z = t \qquad (2.9)
$$

We shall seek the solution of the boundary-value problem, assuming the shape of the wave formation

$$
w_1 = A_0 \sin m_1 z \qquad (2.10)
$$

Thereby, the boundary conditions (2.8) will be satisfied.

From the first equation of the system (2.6) we find

$$F = A_1 e^{n_1 z} + A_2 e^{-n_1 z} + B_1 e^{(n_1+2)z} + B_2 e^{(2-n_1)z} +$$
$$2A_0 Ehr_0 \cot \gamma e^{(1+\nu)z}\left(\Phi_{mn} \sin m_1 z + X_{mn} \cos m_1 z\right) \qquad (2.11)$$

where A_1, A_2, B_1, B_2 are arbitrary constants

$$\left.\begin{aligned}
\Phi_{mn} &= \left[\left(m_1^2 + \nu - \nu^2\right)\varphi_{mn} - 4(\nu - 2\nu^2)m_1^2\psi_{mn}\right] : \left(\varphi_{mn}^2 + 16m_1^2\nu^2\psi_{mn}^2\right)\\
X_{mn} &= m_1\left[(1 - 2\nu)\varphi_{mn} + 4\nu\left(m_1^2 + \nu - \nu^2\right)\psi_{mn}\right] : \left(\varphi_{mn}^2 + 16m_1^2\nu^2\psi_{mn}^2\right)\\
\varphi_{mn} &= \left(m_1^2 + n_1^2\right)^2 - 2\left(1 + \nu^2\right)n_1^2 + 2\left(1 - 3\nu^2\right)m_1^2 + \left(\nu^2 - 1\right)^2\\
\psi_{mn} &= m_1^2 + n_1^2 + 1 - \nu^2
\end{aligned}\right\} \qquad (2.12)$$

Satisfying the conditions (2.9), we arrive at the system of equations for determining A_1, A_2, B_1, and B_2.

As has been shown in reference 1, the initial equations assumed were used only in the case of a thin shell of small length, the buckling of which occurs with the formation of a considerable number of waves along the circumference.

For the approximate determination of the hyperbolic terms in the expression (2.11) we can take

$$\sinh n_1 t \approx \cosh n_1 t$$

especially since, as will be shown later, those terms affect the magnitude of the critical load only insignificantly. In this manner, we find

$$\left.\begin{aligned}
A_1 &\approx -\frac{\Phi_6 + n_1\Phi_5}{2\left(n_1^2 - n_1\right)\sinh n_1 t} e^{(1+\nu)t} \qquad & A_2 &\approx \frac{\Phi_6 - n_1\Phi_5}{n_1^2 + n_1}\\
B_1 &\approx \frac{\Phi_5}{2\left(n_1 + 1\right)\sinh n_1 t} e^{(\nu-1)t} \qquad & B_2 &\approx \frac{\Phi_5}{n_1 - 1}
\end{aligned}\right\} \qquad (2.13)$$

where

$$\left.\begin{aligned} \Phi_5 &= 0.25\left(\Phi_4 - n_1{}^2\chi_{mn}\right) & \Phi_6 &= 0.5\left(2\Phi_2 - n_1{}^2\chi_{mn} - \Phi_4\right) \\ \bar{\Phi}_4 &= (1 + \nu)\Phi_2 + m_1\Phi_1 & \Phi_3 &= (1 + \nu)\Phi_1 - m_1\Phi_2 \\ \Phi_1 &= (1 + \nu)\Phi_{mn} - m_1\chi_{mn} & \Phi_2 &= (1 + \nu)\chi_{mn} + m_1\Phi_{mn} \end{aligned}\right\} \tag{2.14}$$

3. THE CHARACTERISTIC EQUATION

Substituting equations (2.10) and (2.11) into the second equation of system (2.6), we integrate that equation according to the method of Bubnov-Galerkin, multiplying it by $e^{(2+\nu)z}\sin m_1 z\, dz$.

In this manner, we arrive at the characteristic equation

$$P - Q - R + T = 0 \tag{3.1}$$

where

$$\left.\begin{aligned} P &= \frac{m_1\epsilon^2\left[m_1{}^4 + 4(\nu - 1)^2 m_1{}^2 - m_1{}^2 K_{1n} + (1 - \nu)K_{2n} + K_{3n}\right]}{m_1{}^2 + (1 - \nu^2)} \\ Q &= \frac{p_0 m_1}{D'}\left[\frac{0.5m_1{}^2 - 1.5\nu(1 + \nu) - 0.25 + n_1{}^2}{m_1{}^2 + 0.25(1 + 2\nu)^2} + \frac{\lambda\left[m_1{}^2 + 3\nu(1 - \nu) - 0.5\right]}{m_1{}^2 + 0.25(1 - 2\nu)^2}\right] \\ R &= \frac{4\nu}{\left(\tau^{2\nu} - 1\right)}\left\{-\frac{\left(\Phi_6 + n_1\Phi_5\right)e^{2\nu t}}{m_1{}^2 + \left(n_1 + \nu - 1\right)^2} + \frac{\left(\Phi_6 - n_1\Phi_5\right)}{m_1{}^2 + \left(n_1 + \nu - 1\right)^2} + \right. \\ &\qquad \left. \frac{\Phi_5 e^{2\nu t}\left(n_1 + 2\right)}{m_1{}^2 + \left(n_1 + 1 + \nu\right)^2} + \frac{\Phi_5\left(n_1 - 2\right)}{m_1{}^2 + \left(n_1 - 1 - \nu\right)^2}\right\} \\ T &= \frac{m_1\left(\Phi_1 - \Phi_3\right) + \nu\left(\Phi_4 - \Phi_2\right)}{m_1{}^2 + \nu^2} \end{aligned}\right\} \tag{3.2}$$

This equation may be greatly simplified for the wide range of thin shells which satisfy the condition

$$t = \ln\left(1 + \frac{L}{r_0}\right) \leqq 1 \qquad L \leqq 1.72r_0$$

Then we have according to equations (1.2), $m_1^2 \geqq \pi^2$. In addition, for thin shells of small length n_1^2 is much greater than 1. Thereby we have, retaining only the main terms in the expressions of the quantities (2.12), (2.14), and (3.2)

$$\frac{R}{T} \approx \frac{4\nu\left[\sigma n_1^4 - 2\sigma n_1^3 - m_1^2 n_1^2 + 2(1 - \sigma)m_1^4\right]}{m_1^2\left[m_1^2 + (n_1 + \nu + 1)^2\right]\left[m_1^2 \mp (n_1 + \nu - 1)^2\right]}$$

Calculations according to this formula, which take into account the expression P and the solution obtained further on, show that the quantity R in equation (3.1) may be neglected if we permit an error in the magnitude of the critical load of 2 to 3 percent in the direction of increased load. In the remaining terms of the equation we can also neglect the quantities of the order one in comparison with n_1^2. This gives an additional error of 1 to 2 percent.

Let us note that the maximum error is admitted on the boundary of the region where the shell becomes rather long and, for stability, requires transverse reinforcements. With the reduction of t the permissible error rapidly drops to 1 to 2 percent. Thus, we shall determine the critical load from the approximate equation

$$\frac{p_0}{D'}\left(\frac{n_1^2 + 0.5m_1^2}{m_1^2 + 0.25(1 + 2\nu)^2} + \frac{\lambda m_1^2}{m_1^2 + 0.25(1 - 2\nu)^2}\right) =$$

$$\frac{\epsilon^2\left(m_1^2 + n_1^2\right)^2}{m_1^2 + (1 - \nu)^2} + \frac{m_1^4 + \mu_1 m_1^2 + \mu_2}{\left(m_1^2 + n_1^2\right)^2\left(m_1^2 + \nu^2\right)} \tag{3.3}$$

In the special case of longitudinal compression m_1^2 is much greater than 1 and taking the designations (1.3) and (1.4) for $p_0 = 0$ into consideration, we find

$$\frac{T_0(1 + 2\nu)\left(1 - \tau^{(2\nu-1)}\right)}{D'(1 - 2\nu)\left(\tau^{(1+2\nu)} - 1\right)} = \frac{\epsilon^2\left(m_1^2 + n_1^2\right)^2}{m_1^2} + \frac{m_1^2}{\left(m_1^2 + n_1^2\right)^2}$$

Consequently, the critical compressive force is equal to

$$T_0 = \frac{(1 - 2\nu)}{\sqrt{\nu(1 - \nu)}} \frac{\sqrt{KD\left(1 - \sigma^2\right)\left(\tau^{2\nu} - 1\right)\left(1 - \tau^{(2\nu-2)}\right)}}{R_0\left(1 - \tau^{(2\nu-1)}\right)} \tag{3.4}$$

The deviation of T_0 given by the above formula from the corresponding exact formula obtained by Shtaerman does not exceed 4 to 5 percent for $t \leqq 1$.

4. DETERMINATION OF THE CRITICAL LOAD

Using the notation (1.4) and, in addition

$$\delta = \frac{m_1^2 + n_1^2}{m_1} \tag{4.1}$$

we bring equation (3.3) into the form

$$p_0 = \frac{K'}{\lambda_1 - 1 + \delta/m_1}\left(\eta^2\delta^2 + \frac{1}{\delta^2}\right) \tag{4.2}$$

From the condition $\partial p_0/\partial\delta = 0$ we find

$$\eta^2\delta^4 = \left[3 + \frac{2\left(\lambda_1 - 1\right)m_1}{\delta}\right] : \left[1 + 2\left(\lambda_1 - 1\right)\frac{m_1}{\delta}\right] \tag{4.3}$$

By means of simple, but rather tedious calculations we can show that $\partial p_0/\partial m_1$ is greater than 0.

Such a monotonic increase of the quantity p_0 with increasing m_1 can be explained by the fact that the quantities K', λ_1, and η^2 are changed very little by the increase of m_1, owing to the fact that, as can be seen from equation (4.2), for every fixed δ the minimum p_0 is reached at the largest of the values δ/m_1, permitted by the boundary conditions. Therefore

$$m = 1 \qquad m_1 = \pi/t \tag{4.4}$$

From $\lambda_1 = 1$ in equation (4.3) we find

$$\delta = \delta_1 = \sqrt[4]{3} : \sqrt{\eta} \tag{4.5}$$

If $m_1^2 \gg 1$, this solution holds in the case where the prebuckling stress is equal to the ring stress.

In the general case

$$\delta = \frac{\delta_1}{1 + \beta} \tag{4.6}$$

where β satisfies the equation

$$\frac{2\left(1 - \lambda_1\right)m_1}{\delta_1} = \frac{\left[1 - (1 + \beta)^4\right]}{(1 + \beta)\left[1 - \frac{1}{3}(1 + \beta)^4\right]} \tag{4.7}$$

Hence it follows that

$$-0.2 \leqq \beta \leqq 0.2, \text{ if } \quad 0.86 \geqq 2\left(1 - \lambda_1\right)\frac{m_1}{\delta_1} \geqq -2.93 \tag{4.8}$$

Thus we may assume for β the smaller (with regard to the modulus) root of the equation

$$\beta^2\left[9 - 4\left(1 - \lambda_1\right)\frac{m_1}{\delta_1}\right] + \beta\left[6 - 2\left(1 - \lambda_1\right)\frac{m_1}{\delta_1}\right] + 2\left(1 - \lambda_1\right)\frac{m_1}{\delta_1} = 0 \tag{4.9}$$

and the approximate value of the critical pressure is equal to

$$p_{0,k} = 1.31K'\eta^{3/2}m_1\left(1.33 + 2\beta^2\right) : \left[1 + \frac{m_1}{\delta_1}\left(\lambda_1 - 1\right)(1 + \beta)\right] \tag{4.10}$$

In the neighborhood of its minimum, the value p_0 changes slowly; therefore, the critical value of the pressure determined for β (which was found from (4.9)) differs even on the boundaries of the region (4.8) from its value for β which satisfies the condition of minimizing (4.7) by less than 0.6 percent, although the error in the magnitude of β attains 13 percent.

For shells which satisfy the condition

$$0.49 \geqq 2\left(1 - \lambda_1\right)\frac{m_1}{\delta_1} \geqq -0.83 \tag{4.11}$$

that is, for $-0.1 \leqq \beta \leqq 0.1$, admitting a conservative error of about 1 percent, we may put $\beta = 0$ in equation (4.10). In this manner, we arrive at the simple formula

$$p_{0,k} = \frac{1.74K'm_1\eta^{3/2}}{1 + (m_1/\delta_1)(\lambda_1 - 1)} \tag{4.12}$$

Here and in equation (4.10), the quantities K', m_1, η, δ_1, and λ_1 are determined according to formulas (4.4), (4.5), and (1.1) to (1.4).

For $\lambda = 0$, $\lambda_1 = 0.5$ we obtain therefore the following formula for the case of isotropic pressure

$$p_{0,k} = \frac{1.74K'm_1\eta^{3/2}}{1 - 0.5m_1/\delta_1} \tag{4.13}$$

In the special case of a cylindrical shell with the radius R we have

$$\left.\begin{aligned}
&\gamma = 0 \qquad t \approx \frac{L \sin \gamma}{R} \qquad m_1 = \frac{\pi R}{L \sin \gamma} \qquad K' = \frac{2Eh}{R \sin^2\gamma} \\
&\lambda_1 = 0.5 + \lambda \qquad \lambda = \frac{T_0}{p_0 R} \qquad \eta = \frac{h \sin^2\gamma}{R\sqrt{3(1 - \sigma^2)}}
\end{aligned}\right\}$$

According to formula (4.12) we find the magnitude of the critical isotropic pressure in the case of simultaneous action of the axial compressive force T_0:

$$p_0 = \frac{4.85Eh}{L(1 - \sigma^2)^{3/4}}\left(\frac{h}{R}\right)^{3/2} : \left[1 + \frac{1.82\sqrt{hR}}{L(1 - \sigma^2)^{1/4}}\left(\frac{T_0}{p_0 R} - 0.5\right)\right] \tag{4.14}$$

whereby the condition

$$0.265 \geqq (1 - 2\lambda)\frac{\sqrt{hR}}{L} \geqq -0.45 \tag{4.15}$$

must be fulfilled.

Let us note that equation (4.14) is equivalent to the equation

$$\frac{T}{T_{0,m}} + \frac{p_0}{p_{0,m}} = 1 \tag{4.16}$$

where

$$T_{0m} = \frac{2.68Eh^2}{R\sqrt{(1 - \sigma^2)}} \qquad p_{0m} = \frac{4.85Eh}{L(1 - \sigma^2)^{3/4}}\left(\frac{h}{R}\right)^{3/2} : \left[1 - \frac{0.91\sqrt{hR}}{L(1 - \sigma^2)^{1/4}}\right] \tag{4.17}$$

This last equation is approximately satisfied only under condition (4.15). Moreover, it is necessary to take into consideration the case where the shell may lose its stability "with a bang" due to axial compression. Therefore, our formulas ought to be used only when the total axial force $T_0 + \frac{1}{2}p_0R$ is smaller than the lower limit of the critical axial force T_{1H} which, as is known (ref. 6), is determined according to the formula

$$T_{1H} \approx \frac{0.78Eh^2}{R} \tag{4.18}$$

Let, furthermore, $T_0 + \frac{1}{2}p_0R$ be less than or equal to $0.78Eh^2/R$ Then, according to equation (4.16)

$$p_0 \geqq p_{0m}\left(1 - \frac{0.78Eh^2}{RT_{0m}}\right) : \left(1 - \frac{p_{0m}R}{2T_{0m}}\right) \tag{4.19}$$

Consequently,

$$1 - \frac{2T_0}{p_0R} \geqq 2 - \frac{1.56Eh^2\left(1 - p_{0m}R/2T_{0m}\right)}{p_{0m}R^2\left(1 - 0.78Eh^2/RT_{0m}\right)}$$

or, taking equations (4.17) into consideration, assuming $\sigma = 0.3$ and neglecting in the second-order terms hR/L^2 in comparison with unity, we have

$$(1 - 2\lambda)\frac{\sqrt{hR}}{L} \geqq \frac{2.78\sqrt{hR}}{L} - 0.42 > -0.45$$

that is, condition (4.15) is fulfilled.

Formula (4.12) which determines the critical normal pressure in the case of simultaneous action of axial compression, can be greatly simplified under the condition $m_1 \geqq \pi$. For $0.25 \leqq \sigma \leqq 0.33$ (as is usually the case with metals), taking into account new notation, formula (4.12) after easy calculations and neglect of second-order terms, can be put in the form

$$p_{0k} = 4.85 \frac{(2 - \sigma)\left(\tau^{(1-\sigma)} - 1\right)^{1/4}\left(1 - \tau^{-(1+\sigma)}\right)^{3/4}}{\sqrt{1 + \sigma}\left(1 - \sigma^2\right)\left(\tau^{(2-\sigma)} - 1\right)\ln \tau} \frac{E \tan \gamma\left(h/R_0\right)^{5/2}}{[1 + \theta(\lambda - 0.5)]} \tag{4.20}$$

where

$$\tau = 1 + \frac{L}{r_0} \qquad \lambda = \frac{(2 - \sigma)T_0\left(1 - \tau^{-\sigma}\right)}{p_0R_0\sigma\left(\tau^{(2-\sigma)} - 1\right)}$$
$$\theta = \frac{1.82}{\sqrt{1 + \sigma}}\left\{\frac{\left[1 - \tau^{-(1+\sigma)}\right]}{\left[\tau^{(1-\sigma)} - 1\right]}\right\}^{1/4}\sqrt{\frac{h}{R_0}}\,\frac{\tan \gamma}{\ln \tau} \tag{4.21}$$

Let us note that in the change from formula (4.12) to the simplified formula (4.20) we reduced the critical pressure at most by 5 percent (for $t = 1$); but formula (4.12), in turn, was derived from the characteristic equation (3.1), by way of simplifications, which increased the pressure we had been seeking by 2 to 3 percent. In this manner, formula (4.20) ultimately gives the upper limit of the critical pressure, reduced by less than 3 percent.

Moreover, we must not forget that formula (4.12) and notably, also, formula (4.20) are derived for the range of variable shell parameters determined by the inequalities (4.11) which, after simplification, assume the form:

$$0.49 \geqq (1 - 2\lambda)\theta \geqq -0.83 \qquad (4.22)$$

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